Review: Tree search

• Initialize the **frontier** using the **starting state**
• While the frontier is not empty
  – Choose a frontier node to expand according to **search strategy** and take it off the frontier
  – If the node contains the **goal state**, return solution
  – Else **expand** the node and add its children to the frontier

• To handle repeated states:
  – Keep an **explored set**; add each node to the explored set every time you expand it
  – Every time you add a node to the frontier, check whether it already exists in the frontier with a higher path cost, and if yes, replace that node with the new one
Review: Uninformed search strategies

• A search strategy is defined by picking the order of node expansion

• Uninformed search strategies use only the information available in the problem definition
  – Breadth-first search
  – Depth-first search
  – Iterative deepening search
  – Uniform-cost search
Informed search strategies

• Idea: give the algorithm “hints” about the desirability of different states
  – Use an evaluation function to rank nodes and select the most promising one for expansion

• Greedy best-first search
• A* search
Heuristic function

- **Heuristic function** $h(n)$ estimates the cost of reaching goal from node $n$

- Example:
Heuristic for the Romania problem
Greedy best-first search

• Expand the node that has the lowest value of the heuristic function $h(n)$
Greedy best-first search example
Greedy best-first search example
Greedy best-first search example
Greedy best-first search example
Properties of greedy best-first search

- **Complete?**
  - No – can get stuck in loops
Properties of greedy best-first search

- **Complete?**
  No – can get stuck in loops

- **Optimal?**
  No
Properties of greedy best-first search

• Complete?
  No – can get stuck in loops

• Optimal?
  No

• Time?
  Worst case: $O(b^m)$
  Can be much better with a good heuristic

• Space?
  Worst case: $O(b^m)$
How can we fix the greedy problem?
A* search

• Idea: avoid expanding paths that are already expensive
• The evaluation function \( f(n) \) is the estimated total cost of the path through node \( n \) to the goal:

\[
f(n) = g(n) + h(n)
\]

\( g(n) \): cost so far to reach \( n \) (path cost)
\( h(n) \): estimated cost from \( n \) to goal (heuristic)
A* search example
A* search example
A* search example
A* search example
A* search example
A* search example
Another example

Uniform cost search vs. A* search

Admissible heuristics

• An admissible heuristic never overestimates the cost to reach the goal, i.e., it is optimistic

• A heuristic $h(n)$ is admissible if for every node $n$, $h(n) \leq h^*(n)$, where $h^*(n)$ is the true cost to reach the goal state from $n$

• Example: straight line distance never overestimates the actual road distance

• **Theorem:** If $h(n)$ is admissible, A* is optimal
Optimality of A*

• **Theorem:** If $h(n)$ is admissible, A* is optimal (if we don’t do repeated state detection)

• Proof sketch:
  – A* expands all nodes for which $f(n) \leq C^*$, i.e., the estimated path cost to the goal is less than or equal to the actual path cost to the first goal encountered
  – When we reach the goal node, all the other nodes remaining on the frontier have estimated path costs to the goal that are at least as big as $C^*$
  – Because we are using an admissible heuristic, the true path costs to the goal for these nodes cannot be less than $C^*$
A* gone wrong?

State space graph

Search tree

Source: Berkeley CS188x
Consistency of heuristics

- **Consistency**: Stronger than admissibility
- **Definition**:
  \[ \text{cost}(A \text{ to } C) + h(C) \geq h(A) \]
  \[ \text{cost}(A \text{ to } C) \geq h(A) - h(C) \]
  \[ \text{real cost} \geq \text{cost implied by heuristic} \]
- **Consequences**:
  - The f value along a path never decreases
  - A* graph search is optimal

Source: [Berkeley CS188x](https://www.cs.berkeley.edu/courses/cs188x.html)
Optimality of A*

• Tree search (i.e., search without repeated state detection):
  – A* is optimal if heuristic is *admissible* (and non-negative)

• Graph search (i.e., search with repeated state detection)
  – A* optimal if heuristic is *consistent*

• Consistency implies admissibility
  – In general, most natural admissible heuristics tend to be consistent, especially if from relaxed problems

Source: Berkeley CS188x
Optimality of A*

• A* is *optimally efficient* – no other tree-based algorithm that uses the same heuristic can expand fewer nodes and still be guaranteed to find the optimal solution
  – Any algorithm that does not expand all nodes with \( f(n) \leq C^* \) risks missing the optimal solution
Properties of A*

- **Complete?**
  Yes – unless there are infinitely many nodes with $f(n) \leq C^*$

- **Optimal?**
  Yes

- **Time?**
  Number of nodes for which $f(n) \leq C^*$ (exponential)

- **Space?**
  Exponential
Designing heuristic functions

- Heuristics for the 8-puzzle
  
  \[ h_1(n) = \text{number of misplaced tiles} \]
  
  \[ h_2(n) = \text{total Manhattan distance (number of squares from desired location of each tile)} \]

- \( h_1(\text{start}) = 8 \)
  
  \[ h_2(\text{start}) = 3+1+2+2+2+3+3+2 = 18 \]

- Are \( h_1 \) and \( h_2 \) admissible?
Heuristics from relaxed problems

• A problem with fewer restrictions on the actions is called a relaxed problem
• The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
• If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution
• If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution
Heuristics from subproblems

- Let $h_3(n)$ be the cost of getting a subset of tiles (say, 1,2,3,4) into their correct positions.
- Can precompute and save the exact solution cost for every possible subproblem instance – *pattern database*.
Dominance

• If \( h_1 \) and \( h_2 \) are both admissible heuristics and \( h_2(n) \geq h_1(n) \) for all \( n \), (both admissible) then \( h_2 \) dominates \( h_1 \)

• Which one is better for search?
  – A* search expands every node with \( f(n) < C^* \) or \( h(n) < C^* - g(n) \)
  – Therefore, A* search with \( h_1 \) will expand more nodes
Dominance

• Typical search costs for the 8-puzzle (average number of nodes expanded for different solution depths):

• $d=12$  
  
  $\text{IDS} = 3,644,035$ nodes  
  
  $A^*(h_1) = 227$ nodes  
  
  $A^*(h_2) = 73$ nodes

• $d=24$  
  
  $\text{IDS} \approx 54,000,000,000$ nodes  
  
  $A^*(h_1) = 39,135$ nodes  
  
  $A^*(h_2) = 1,641$ nodes
Combining heuristics

• Suppose we have a collection of admissible heuristics $h_1(n), h_2(n), \ldots, h_m(n)$, but none of them dominates the others.

• How can we combine them?

\[
h(n) = \max\{h_1(n), h_2(n), \ldots, h_m(n)\}
\]
Weighted A* search

- **Idea:** speed up search at the expense of optimality
- Take an admissible heuristic, “inflate” it by a multiple $\alpha > 1$, and then perform A* search as usual
- Fewer nodes tend to get expanded, but the resulting solution may be suboptimal (its cost will be at most $\alpha$ times the cost of the optimal solution)
Example of weighted A* search

Heuristic: 5 * Euclidean distance from goal
Example of weighted A* search

Heuristic: 5 * Euclidean distance from goal

Compare: Exact A*
Additional pointers

- Interactive path finding demo
- Variants of A* for path finding on grids
<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Complete?</th>
<th>Optimal?</th>
<th>Time complexity</th>
<th>Space complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>BFS</td>
<td>Yes</td>
<td>If all step costs are equal</td>
<td>$O(b^d)$</td>
<td>$O(b^d)$</td>
</tr>
<tr>
<td>DFS</td>
<td>No</td>
<td>No</td>
<td>$O(b^m)$</td>
<td>$O(bm)$</td>
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<tr>
<td>IDS</td>
<td>Yes</td>
<td>If all step costs are equal</td>
<td>$O(b^d)$</td>
<td>$O(bd)$</td>
</tr>
<tr>
<td>UCS</td>
<td>Yes</td>
<td>Yes</td>
<td>Number of nodes with $g(n) \leq C^*$</td>
<td></td>
</tr>
<tr>
<td>Greedy</td>
<td>No</td>
<td>No</td>
<td>Worst case: $O(b^m)$&lt;br&gt;Best case: $O(bd)$</td>
<td></td>
</tr>
<tr>
<td>A*</td>
<td>Yes</td>
<td>Yes (if heuristic is admissible)</td>
<td>Number of nodes with $g(n)+h(n) \leq C^*$</td>
<td></td>
</tr>
</tbody>
</table>