# Review: Tree search

- Initialize the **frontier** using the **starting state**
- While the frontier is not empty
  - Choose a frontier node to expand according to **search strategy** and take it off the frontier
  - If the node contains the **goal state**, return solution
  - Else **expand** the node and add its children to the frontier

- To handle repeated states:
  - Keep an **explored set**; add each node to the explored set every time you expand it
  - Every time you add a node to the frontier, check whether it already exists in the frontier with a higher path cost, and if yes, replace that node with the new one

# Review: Uninformed search strategies

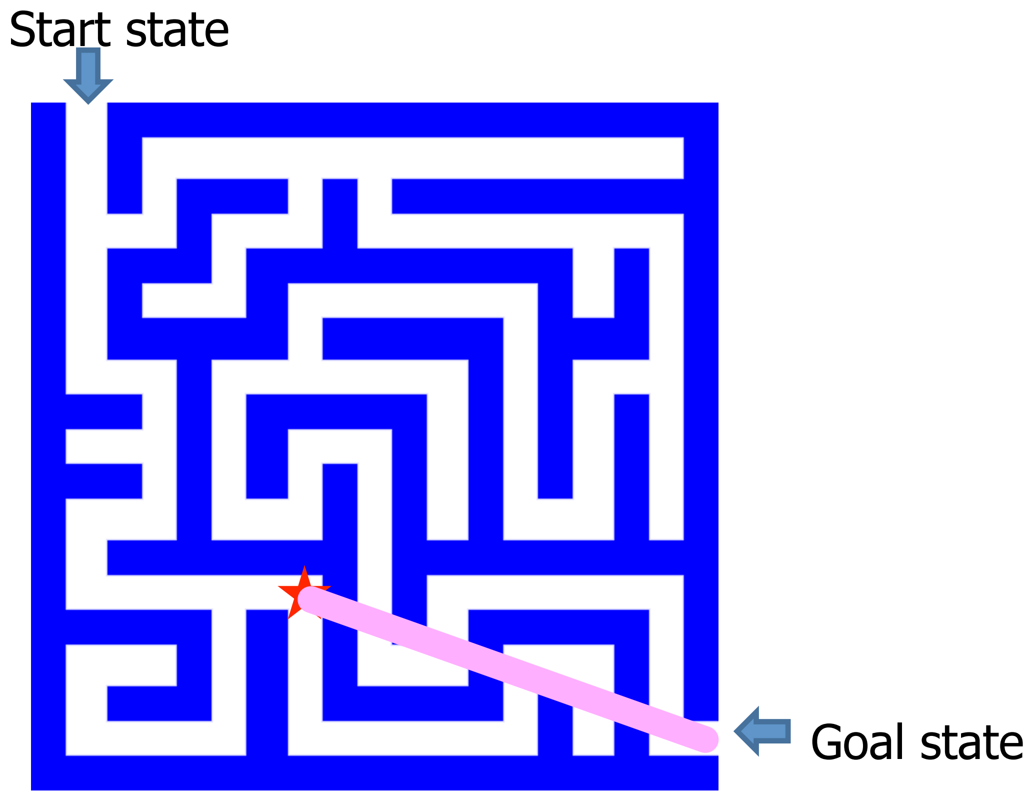
- Breadth-first search
- Depth-first search
- Iterative deepening search
- Uniform-cost search

# Informed search strategies (Sections 3.5-3.6)

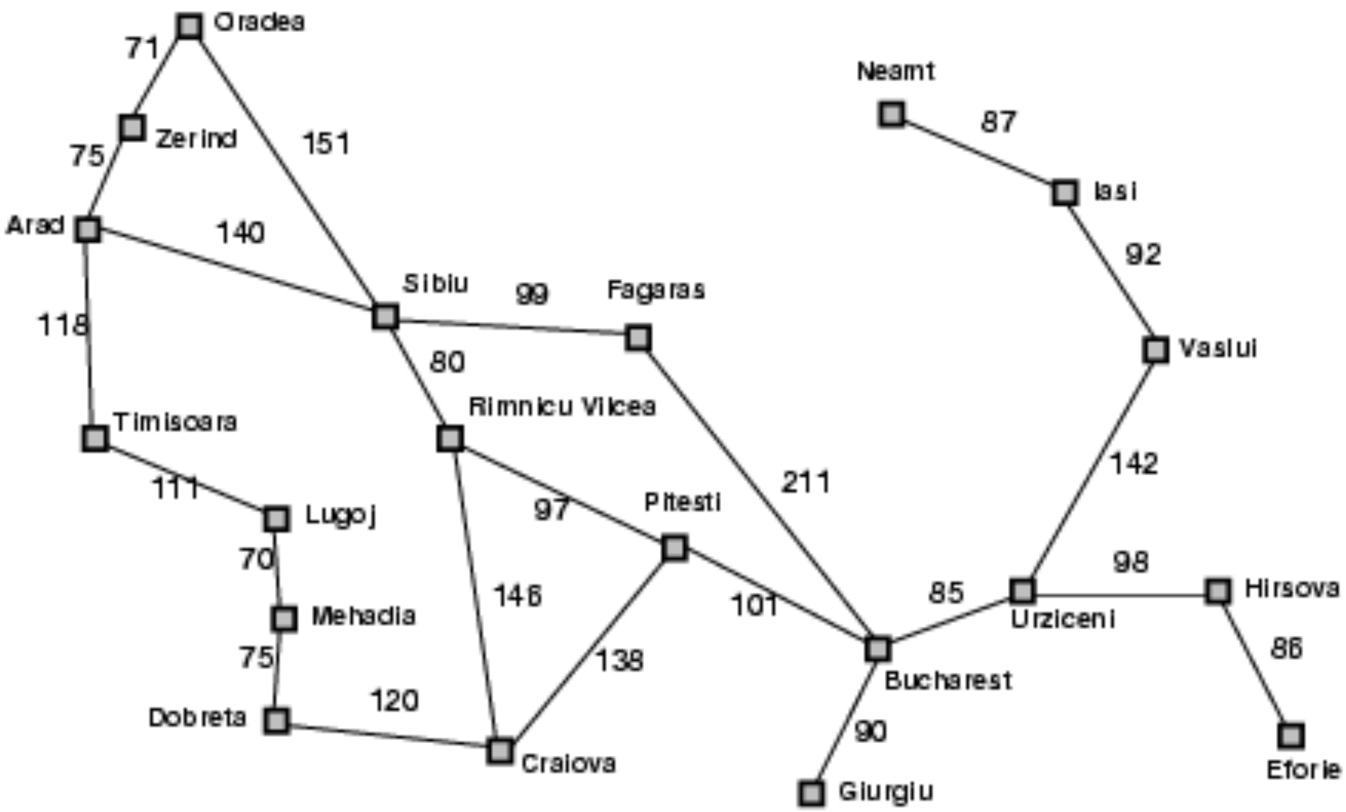
- Idea: give the algorithm "hints" about the desirability of different states
  - Use an ***evaluation function*** to rank nodes and select the most promising one for expansion
- Greedy best-first search
- A* search

# Heuristic function

- **Heuristic function** *h*(*n*) estimates the cost of reaching goal from node *n*
- Example:



# Heuristic for the Romania problem




Straight–line distance to Bucharest

| | |
|---|---|
| **Arad** | 366 |
| **Bucharest** | 0 |
| **Craiova** | 160 |
| **Dobreta** | 242 |
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| **Timisoara** | 329 |
| **Urziceni** | 80 |
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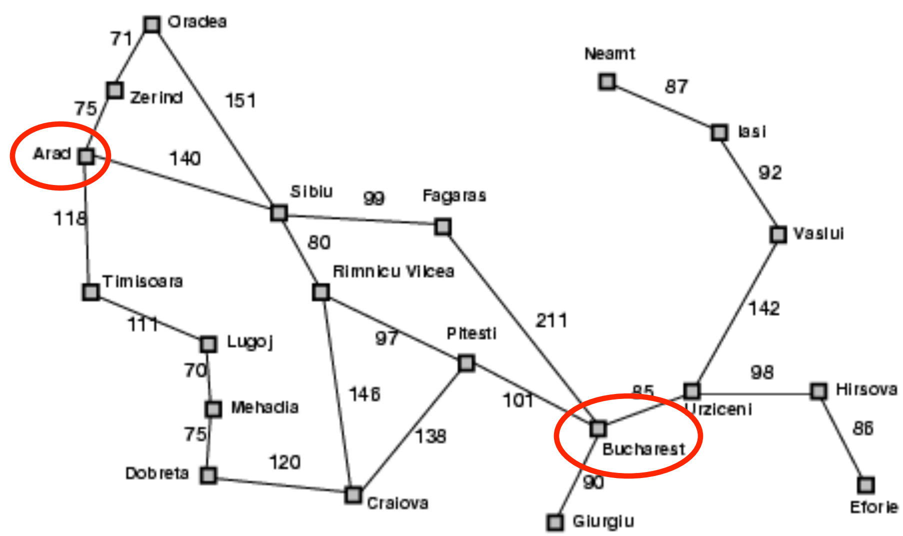
| **Zerind** | 374 |

# Greedy best-first search

- Expand the node that has the lowest value of the heuristic function $h(n)$

# Greedy best-first search example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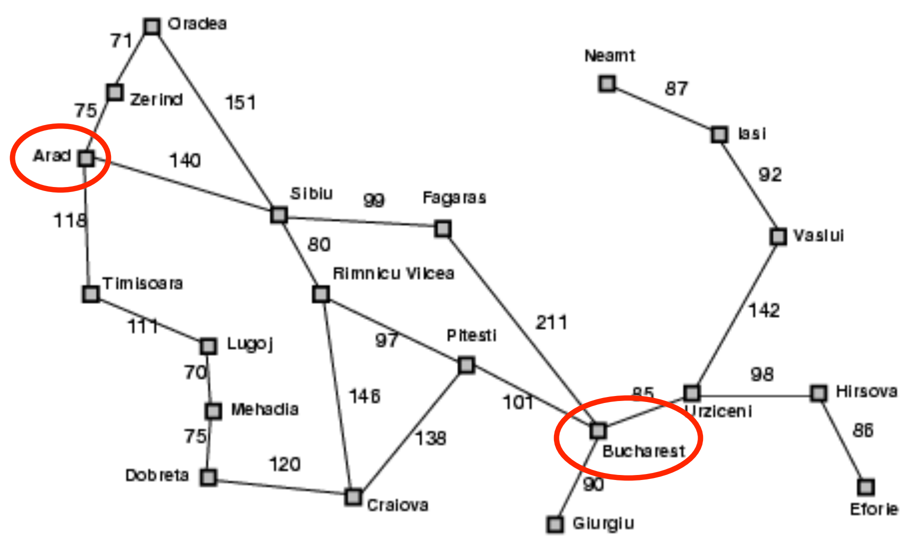
# Greedy best-first search example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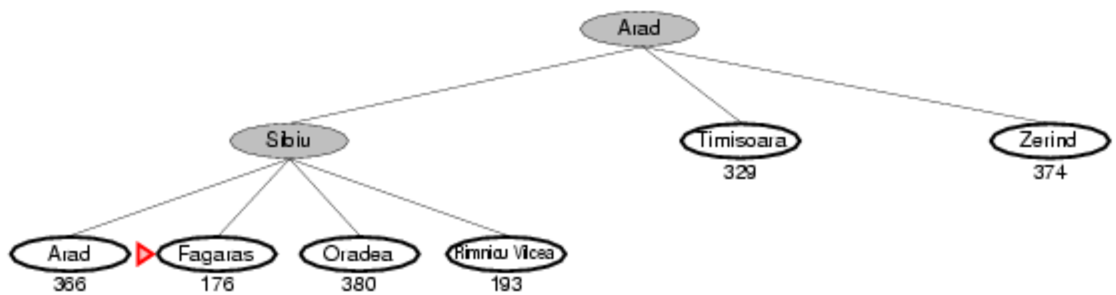
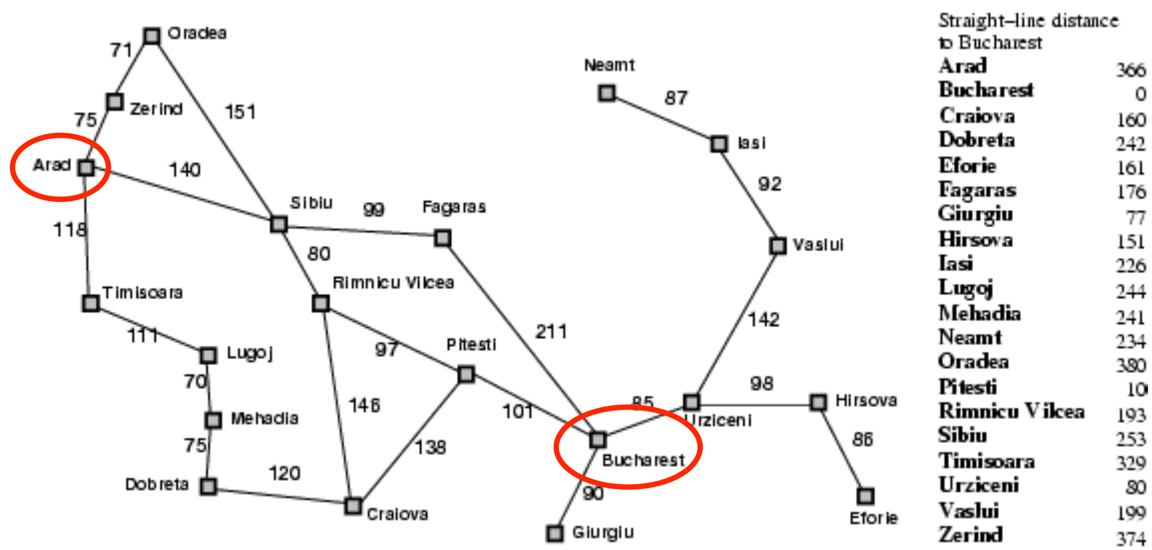
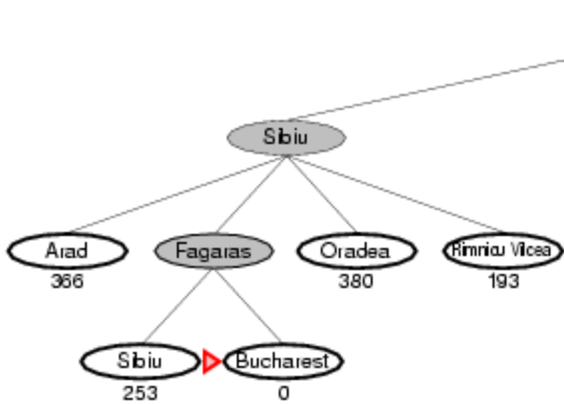
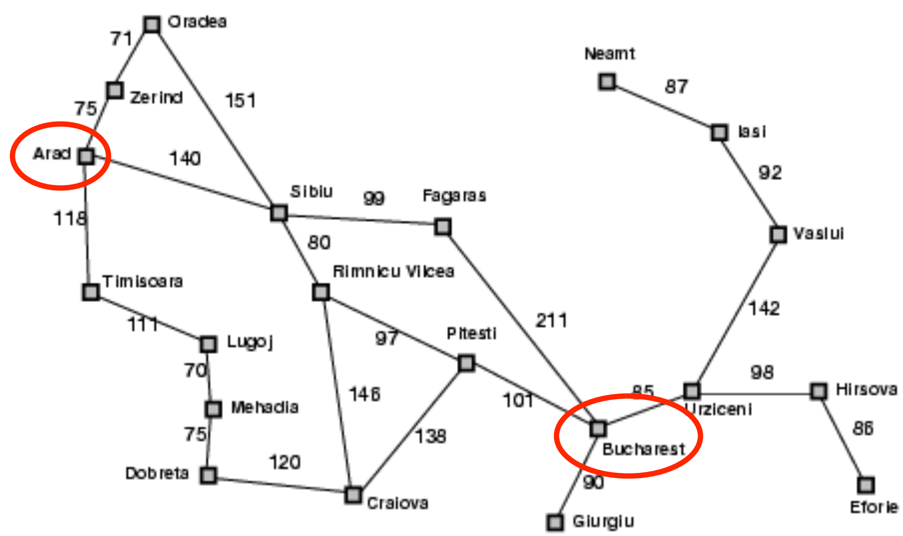
# Greedy best-first search example



# Greedy best-first search example

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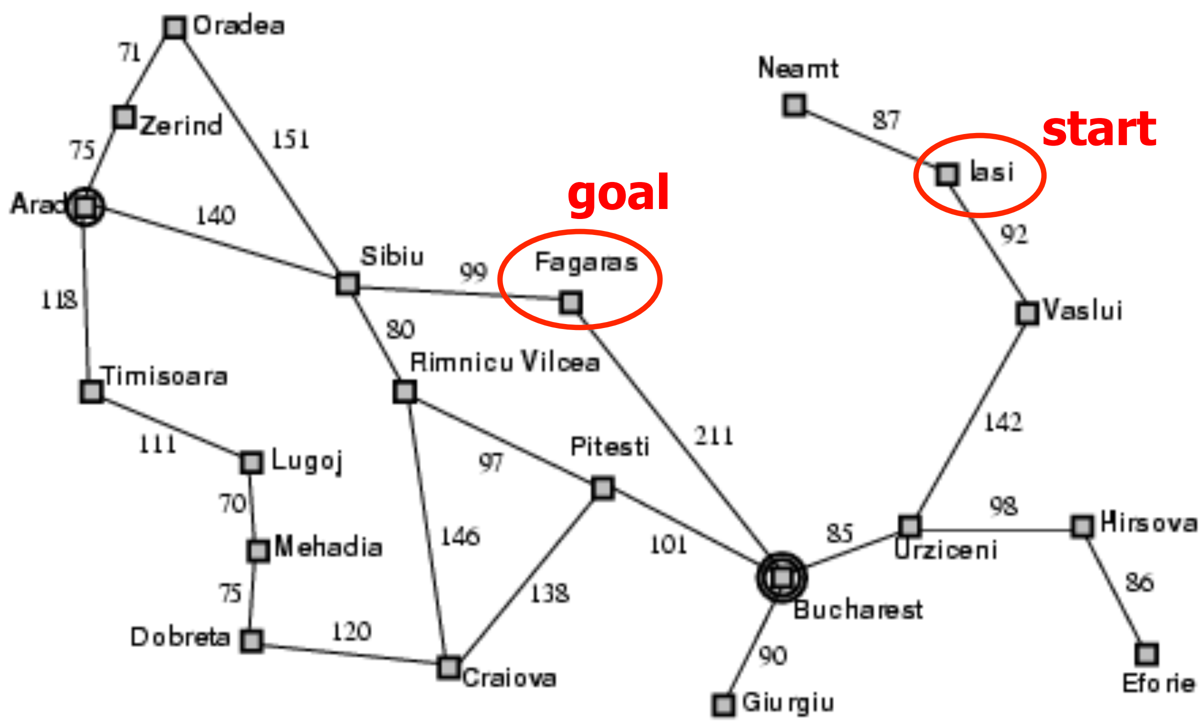
# Properties of greedy best-first search

- **Complete?**

  No – can get stuck in loops



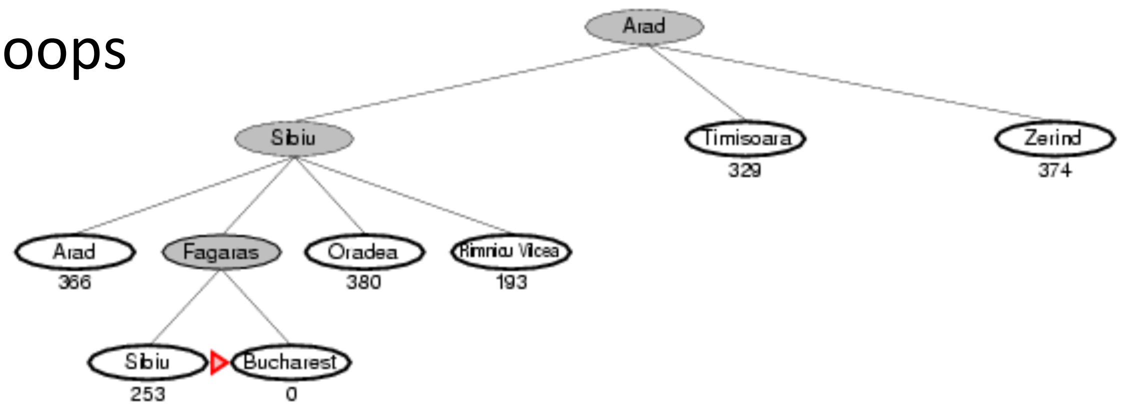
# Properties of greedy best-first search

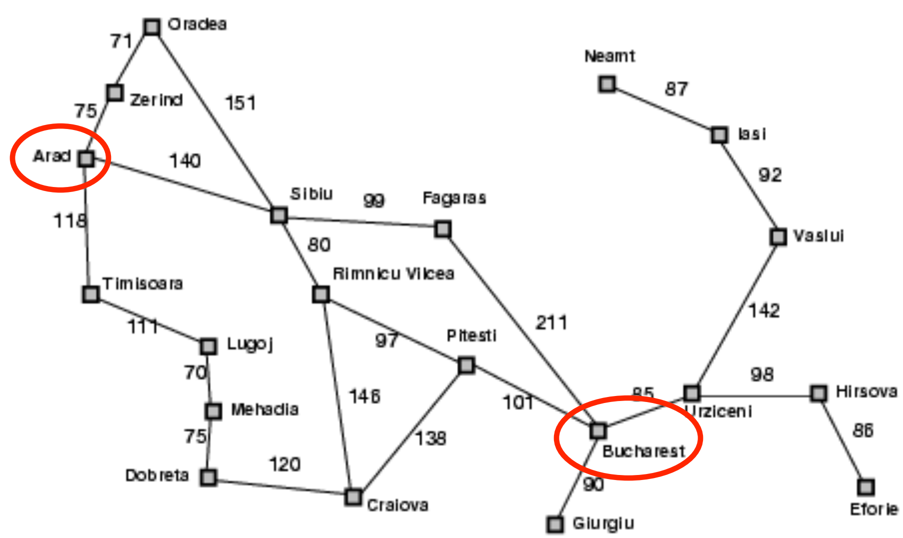
- **Complete?**

  No – can get stuck in loops

- **Optimal?**

  No

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# Properties of greedy best-first search

- **Complete?**

  No – can get stuck in loops

- **Optimal?**

  No

- **Time?**

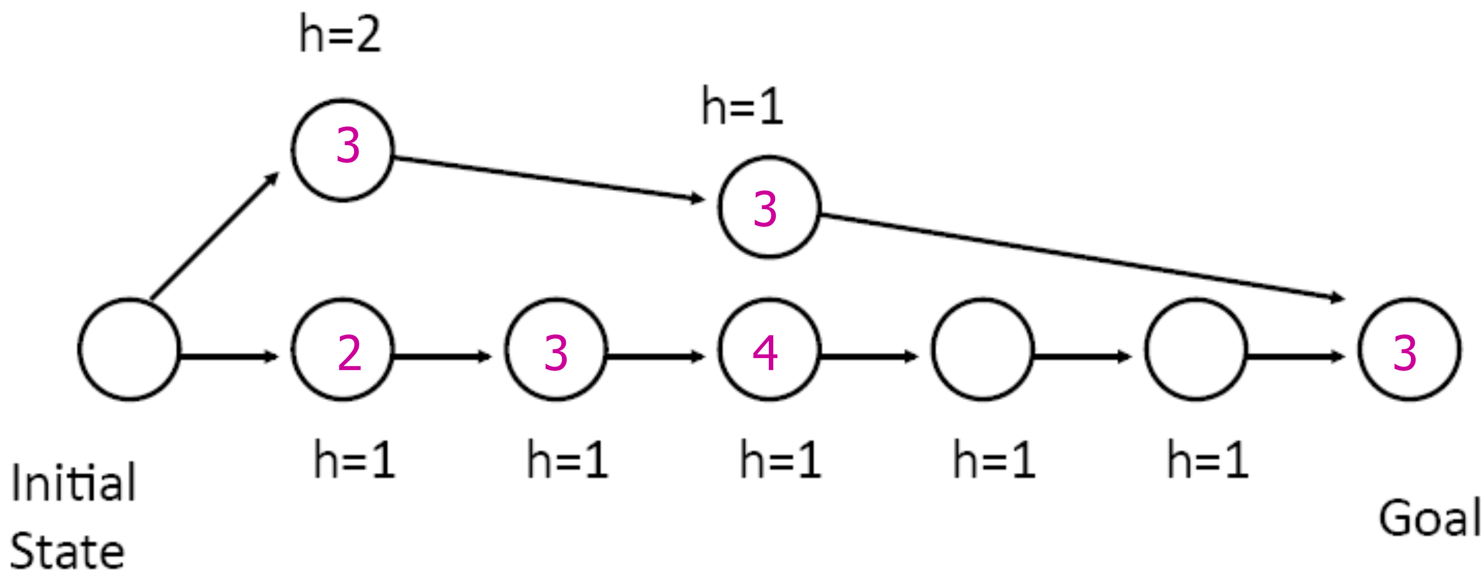
  Worst case: $O(b^m)$

  Can be much better with a good heuristic

- **Space?**

  Worst case: $O(b^m)$

# How can we fix the greedy problem?




- How about keeping track of the distance already traveled in addition to the distance remaining?

# $A^*$ search

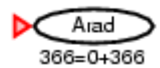
- Idea: avoid expanding paths that are already expensive
- The **evaluation function** $f(n)$ is the estimated total cost of the path through node $n$ to the goal:

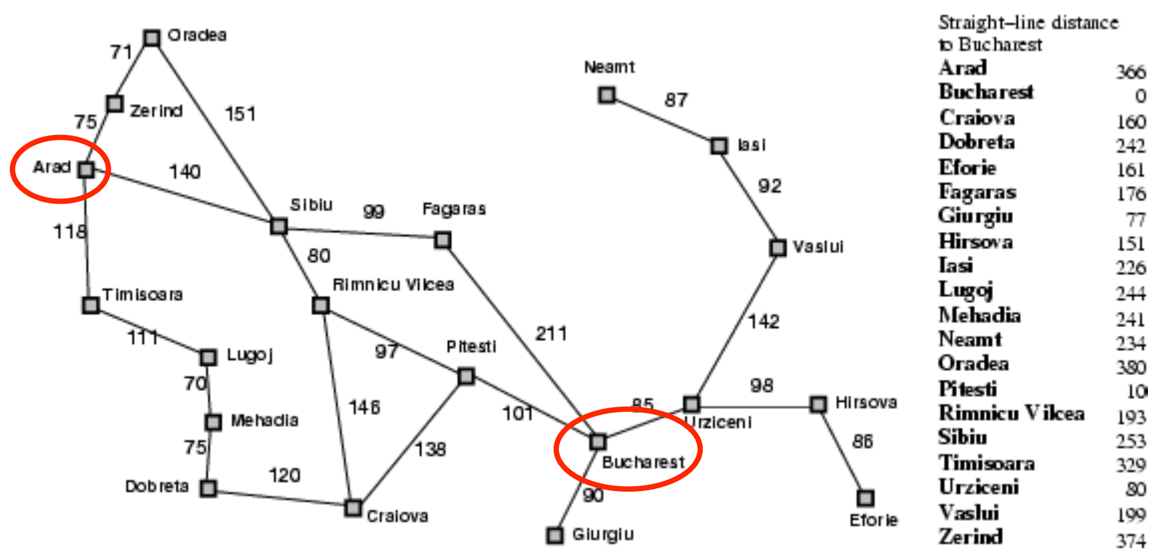
$$f(n) = g(n) + h(n)$$

$g(n)$: cost so far to reach $n$ (path cost)

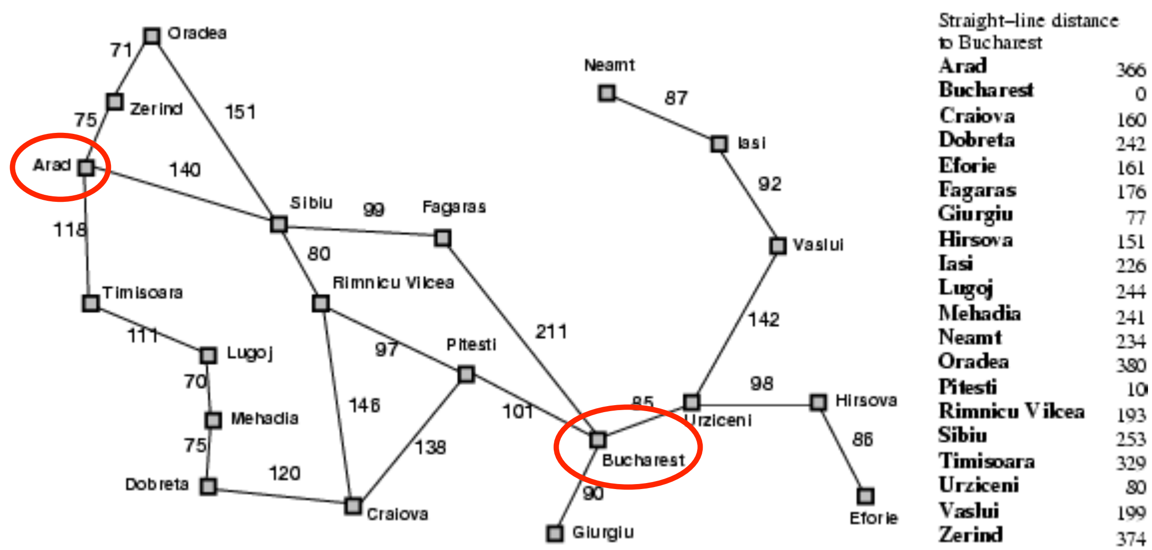
$h(n)$: estimated cost from $n$ to goal (heuristic)

# A* search example



# A* search example

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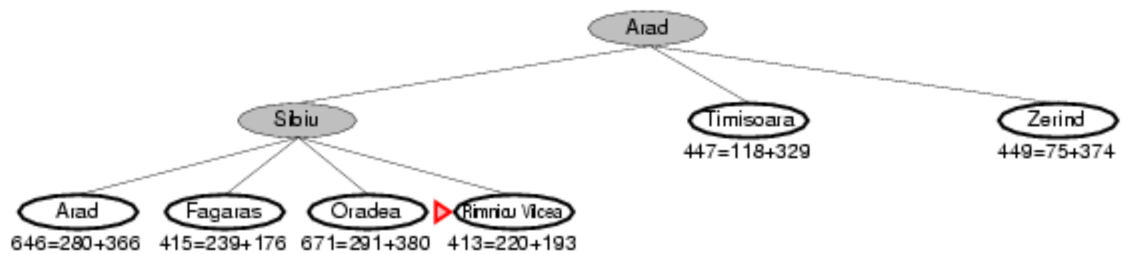


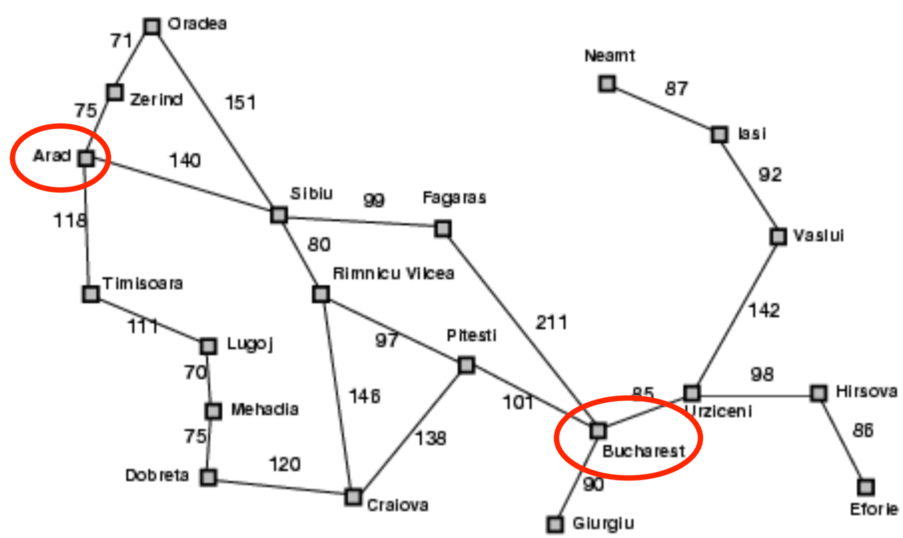


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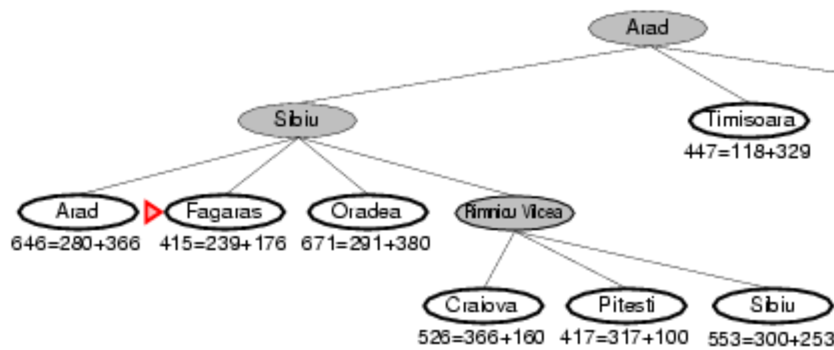
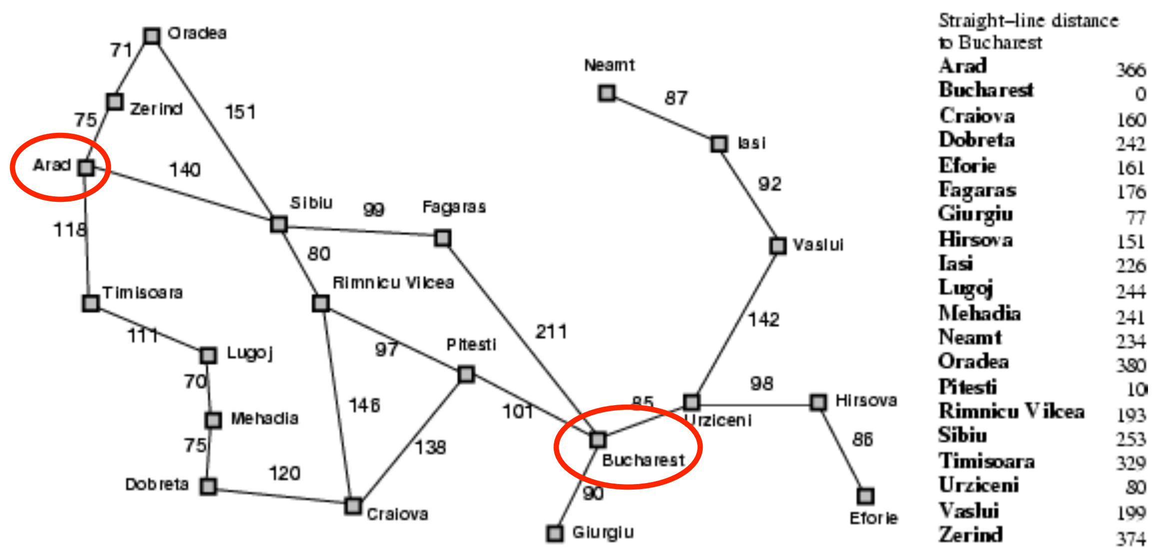
# A* search example



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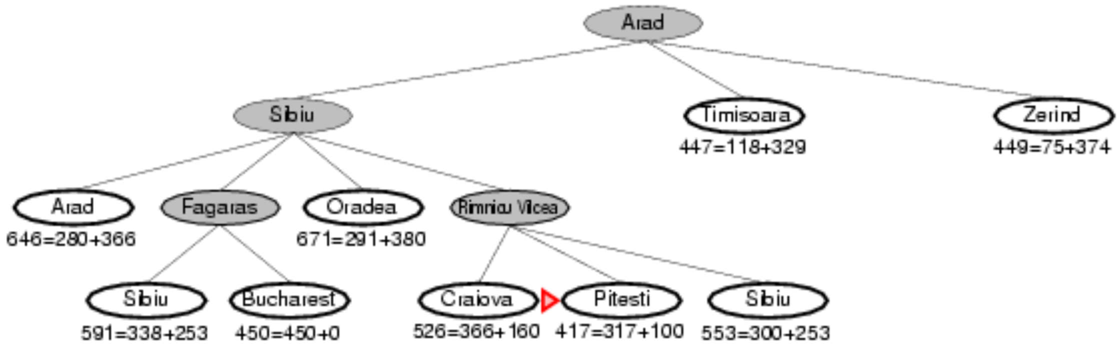
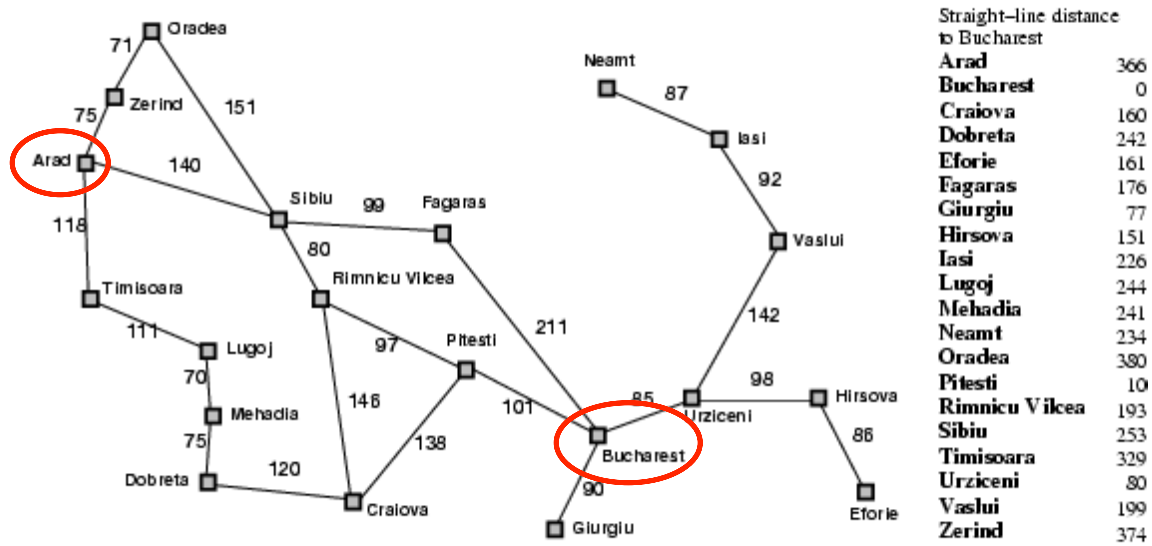
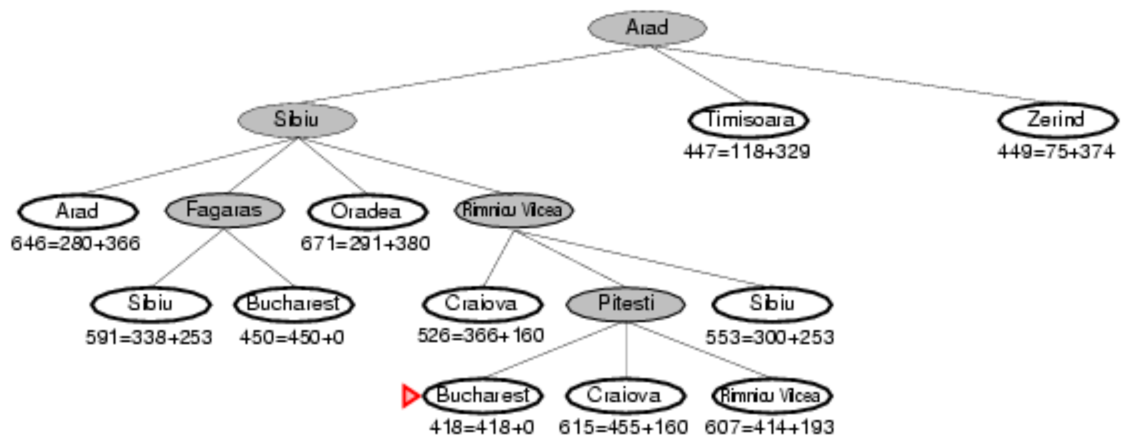
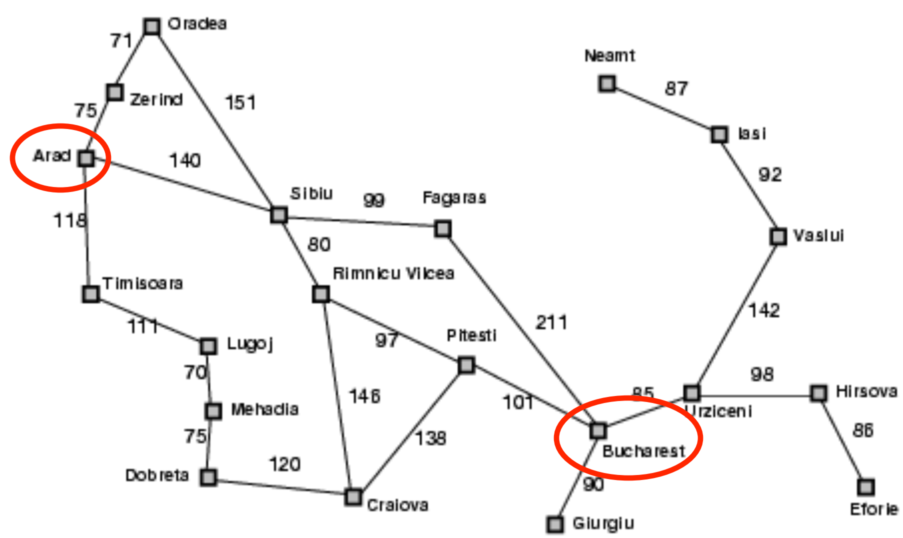


# A$^*$ search example

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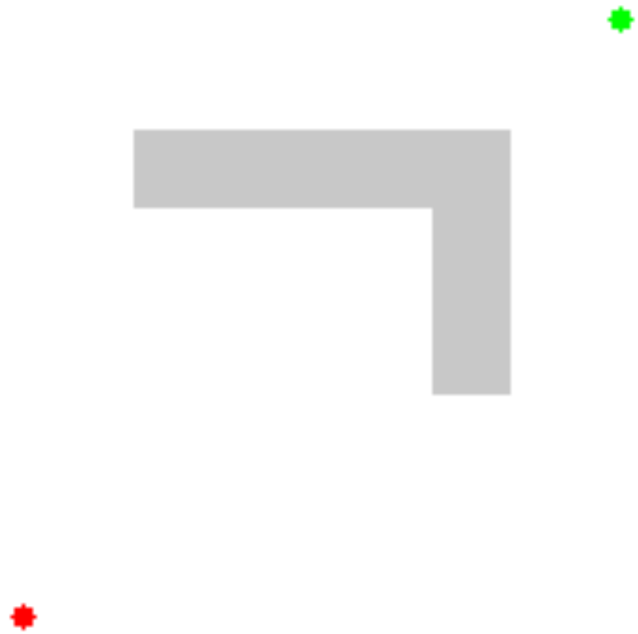
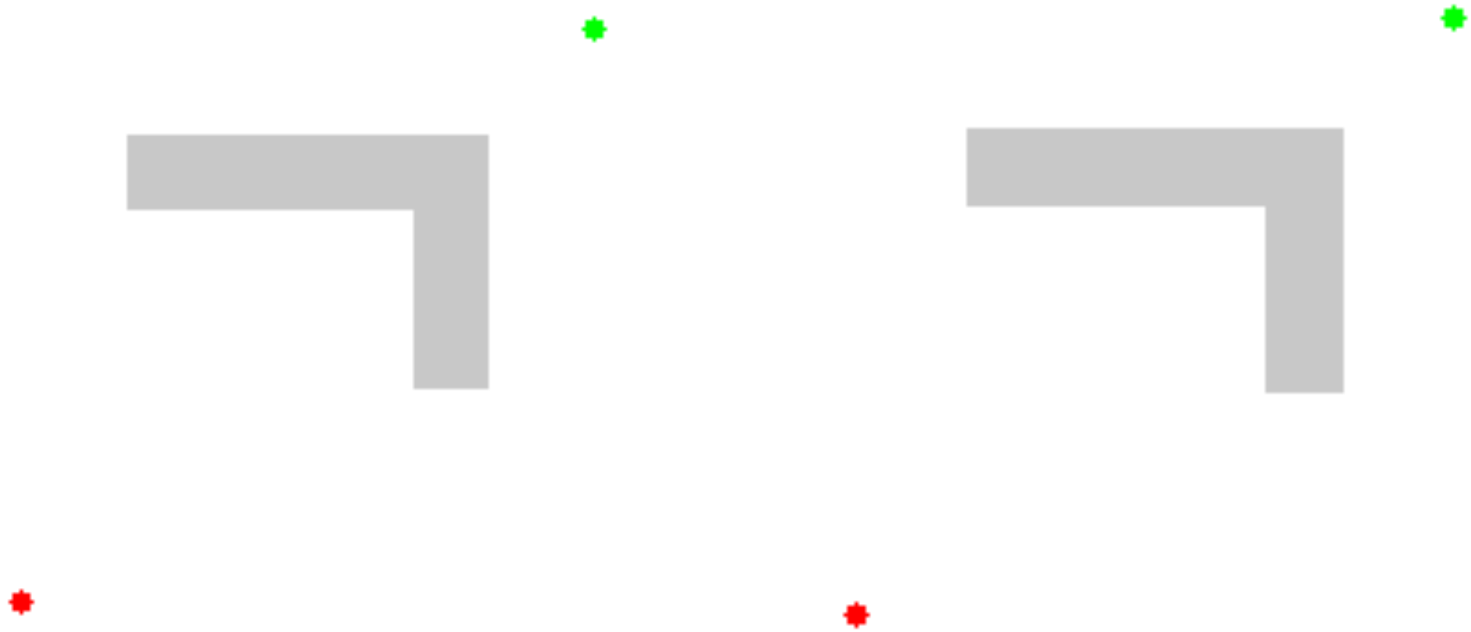
# Another example



Source: Wikipedia

# Uniform cost search vs. A* search



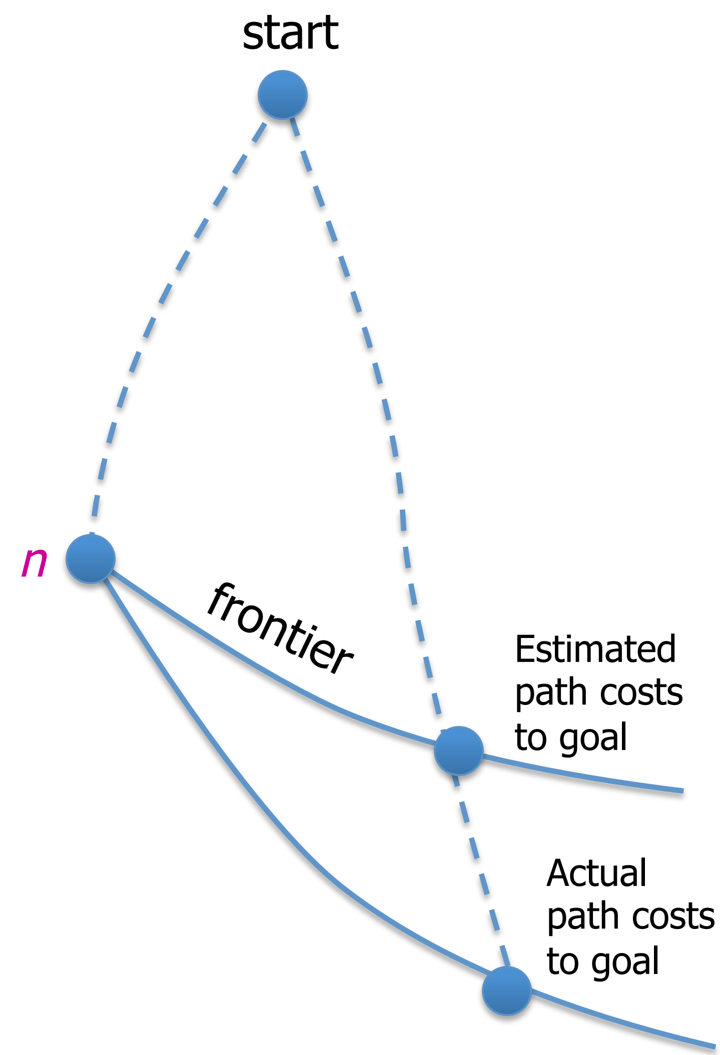
Source: [Wikipedia]

# Admissible heuristics

- An admissible heuristic never overestimates the cost to reach the goal, i.e., it is optimistic
- A heuristic $h(n)$ is admissible if for every node $n$, $h(n) \leq h^*(n)$, where $h^*(n)$ is the true cost to reach the goal state from $n$
- Example: straight line distance never overestimates the actual road distance
- **Theorem:** If $h(n)$ is admissible, $A^*$ is optimal

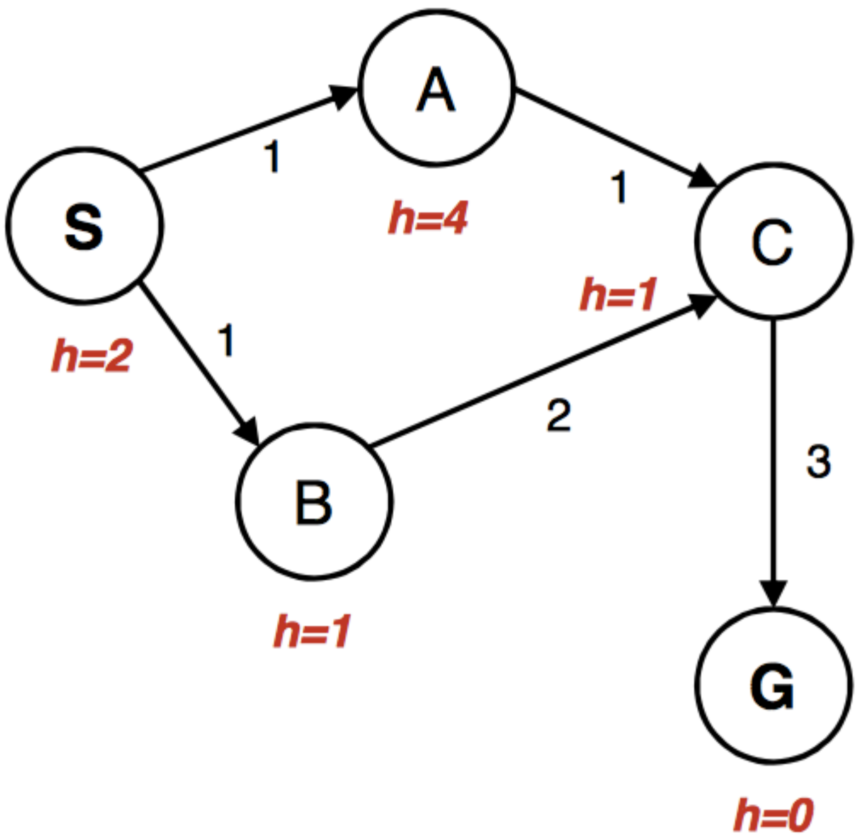
# Optimality of A*

- **Theorem:** If the heuristic is admissible, A* *without repeated state detection* is optimal
- Proof sketch:
  - Let *C** be the evaluation function value (*actual* path cost) of the first goal node we select for expansion
  - Then all the other nodes on the frontier have *estimated* path costs to the goal that are at least as big as *C**
  - Because we are using an admissible heuristic, the *true* path costs to the goal for those nodes cannot be less than *C**

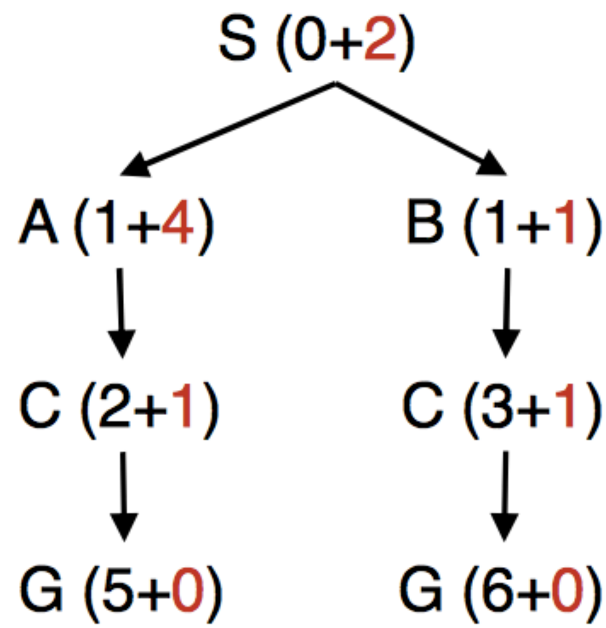


# A* gone wrong?

State space graph

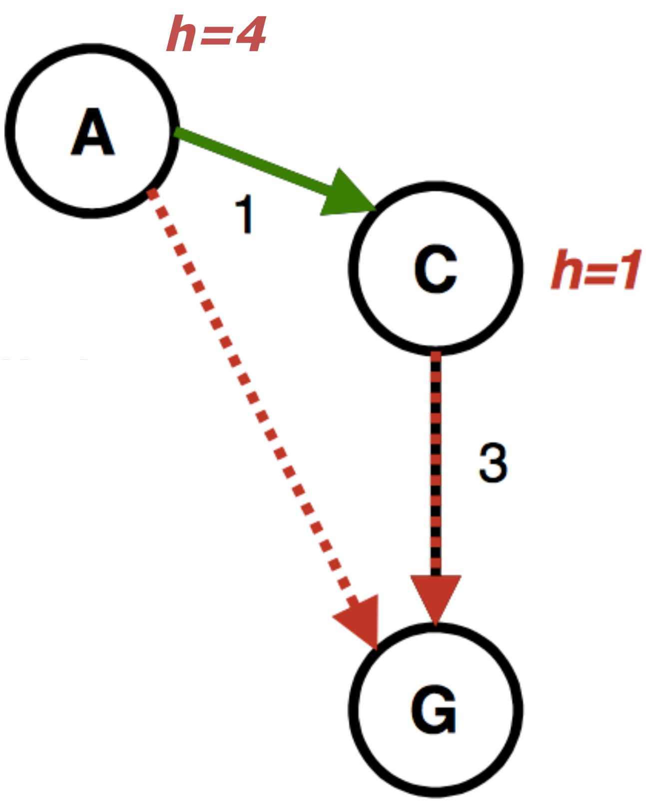



Search tree




Source: [Berkeley CS188x]()

# Consistency of heuristics




- Consistency: Stronger than admissibility
- Definition:

  cost(A to C) + h(C) ≥ h(A)

  cost(A to C) ≥ h(A) - h(C)

  real cost ≥ cost implied by heuristic
- Consequences:
  - The f value along a path never decreases
  - A* graph search is optimal

Source: [Berkeley CS188x]

# Optimality of A*

- **Tree search** (i.e., search without repeated state detection):
  - A* is optimal if heuristic is ***admissible*** (and non-negative)
- **Graph search** (i.e., search with repeated state detection)
  - A* optimal if heuristic is ***consistent***
- Consistency implies admissibility
  - In general, most natural admissible heuristics tend to be consistent, especially if they come from relaxed problems

Source: [Berkeley CS188x]

# Optimality of A*

- A* is *optimally efficient* – no other tree-based algorithm that uses the same heuristic can expand fewer nodes and still be guaranteed to find the optimal solution
  - A* expands all nodes for which $f(n) \le C^*$. Any algorithm that does not risks missing the optimal solution

# Properties of A*

- **Complete?**

  Yes – unless there are infinitely many nodes with $f(n) \leq C^*$

- **Optimal?**

  Yes

- **Time?**

  Number of nodes for which $f(n) \leq C^*$ (exponential)

- **Space?**

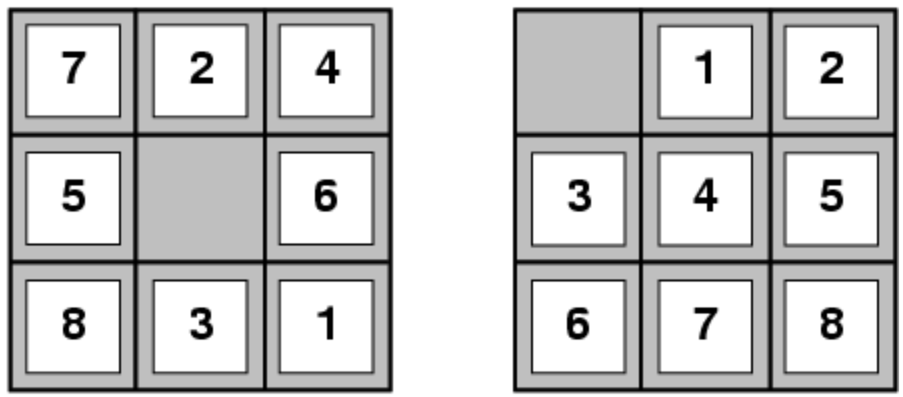
  Exponential

# Designing heuristic functions

- Heuristics for the 8-puzzle

  $h_1(n)$ = number of misplaced tiles

  $h_2(n)$ = total Manhattan distance (number of squares from desired location of each tile)




Start State Goal State

$h_1(\text{start}) = 8$

$h_2(\text{start}) = 3+1+2+2+2+3+3+2 = 18$

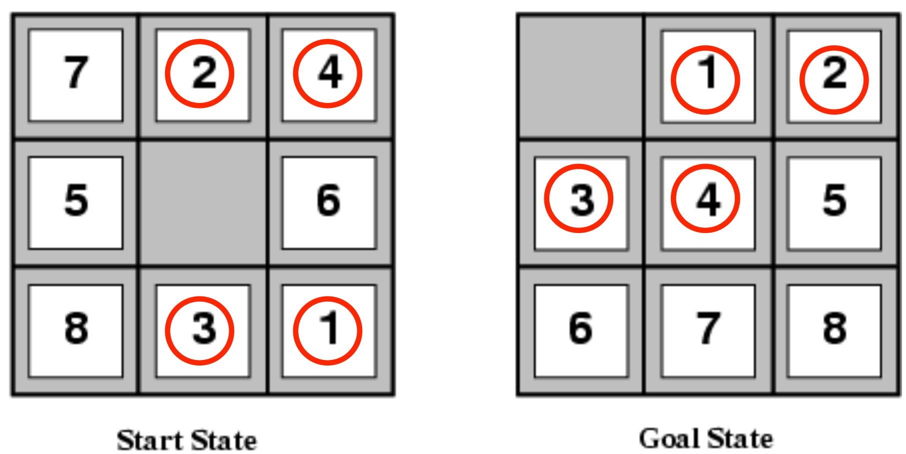
- Are $h_1$ and $h_2$ admissible?

# Heuristics from relaxed problems

- A problem with fewer restrictions on the actions is called a relaxed problem
- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
- If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution
- If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution

# Heuristics from subproblems

- Let $h_3(n)$ be the cost of getting a subset of tiles (say, 1,2,3,4) into their correct positions
- Can precompute and save the exact solution cost for every possible subproblem instance – *pattern database*



# Dominance

- If $h_1$ and $h_2$ are both admissible heuristics and $h_2(n) \geq h_1(n)$ for all $n$, (both admissible) then $h_2$ **dominates** $h_1$
- Which one is better for search?
  - A* search expands every node with $f(n) < C^*$ or $h(n) < C^* - g(n)$
  - Therefore, A* search with $h_1$ will expand more nodes

# Dominance

- Typical search costs for the 8-puzzle (average number of nodes expanded for different solution depths):

- $d$=12 IDS = 3,644,035 nodes
  $A^*(h_1)$ = 227 nodes
  $A^*(h_2)$ = 73 nodes

- $d$=24 IDS ≈ 54,000,000,000 nodes
  $A^*(h_1)$ = 39,135 nodes
  $A^*(h_2)$ = 1,641 nodes

# Combining heuristics

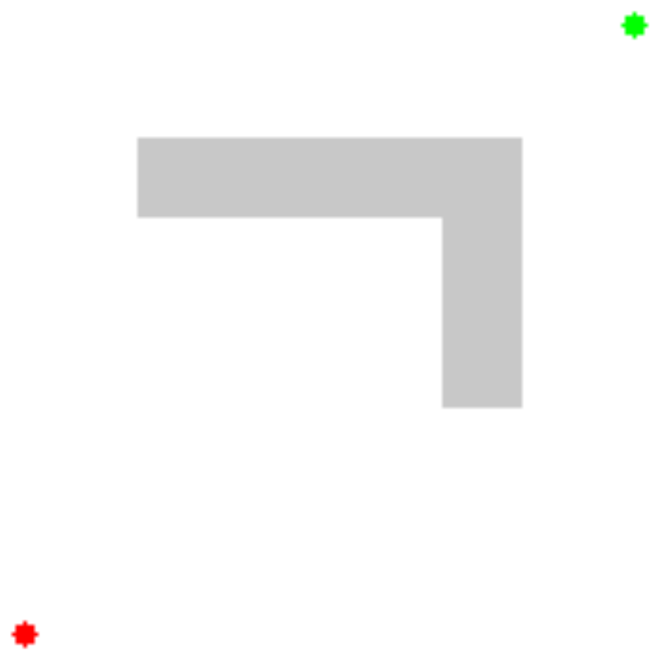
- Suppose we have a collection of admissible heuristics $h_1(n)$, $h_2(n)$, …, $h_m(n)$, but none of them dominates the others
- How can we combine them?

$$h(n) = \max\{h_1(n), h_2(n), \ldots, h_m(n)\}$$

# Weighted A* search

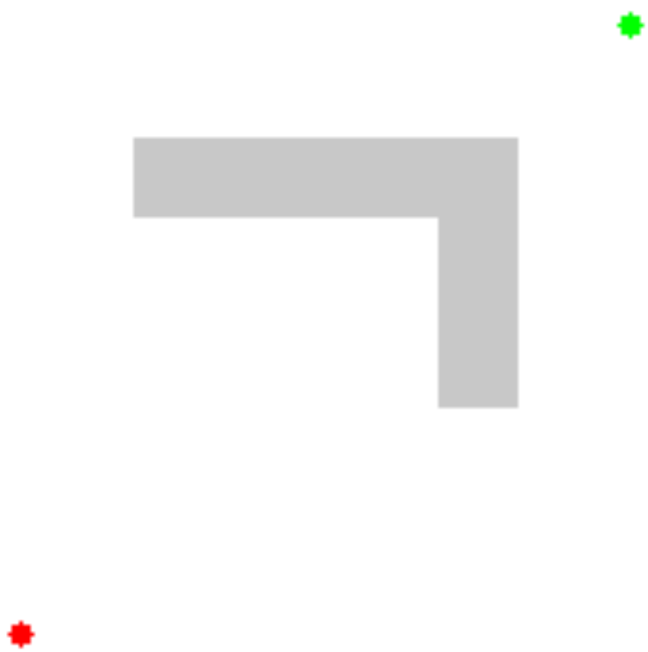
- **Idea:** speed up search at the expense of optimality
- Take an admissible heuristic, “inflate” it by a multiple $\alpha > 1$, and then perform A* search as usual
- Fewer nodes tend to get expanded, but the resulting solution may be suboptimal (its cost will be at most $\alpha$ times the cost of the optimal solution)

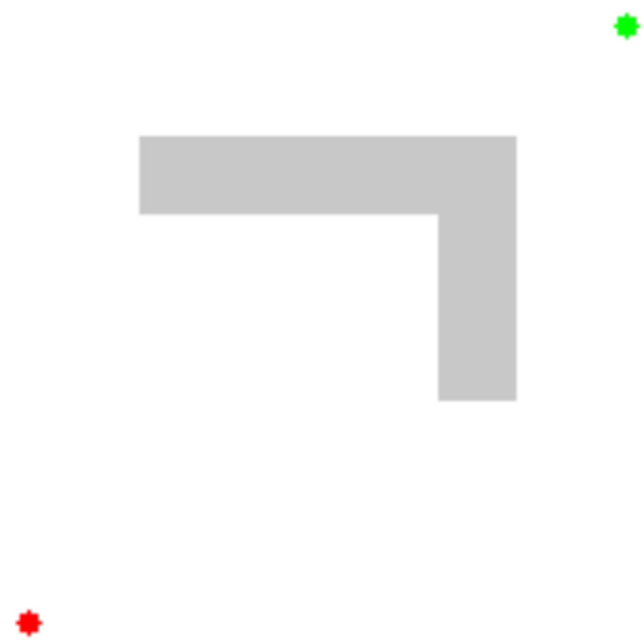
# Example of weighted A* search



Heuristic: 5 * Euclidean distance from goal
Source: Wikipedia

# Example of weighted A* search





Heuristic: 5 * Euclidean distance from goal
Source: [Wikipedia](Wikipedia)

Compare: Exact A*

# Additional pointers

- Interactive path finding demo
- Variants of A* for path finding on grids

# All search strategies

| Algorithm | Complete? | Optimal? | Time complexity | Space complexity |
|---|---|---|---|---|
| **BFS** | Yes | If all step costs are equal | $O(b^d)$ | $O(b^d)$ |
| **DFS** | No | No | $O(b^m)$ | $O(bm)$ |
| **IDS** | Yes | If all step costs are equal | $O(b^d)$ | $O(bd)$ |
| **UCS** | Yes | Yes | Number of nodes with $g(n) \leq C^*$ | |
| **Greedy** | No | No | Worst case: $O(b^m)$<br>Best case: $O(bd)$ | |
| **A*** | Yes | Yes (if heuristic is admissible) | Number of nodes with $g(n)+h(n) \leq C^*$ | |

# A note on the complexity of search

- We said that the worst-case complexity of search is exponential in the length of the solution path
  – But the length of the solution path can be exponential in the number of “objects” in the problem!
- Example: towers of Hanoi