Game theory
(Sections 17.5-17.6)
Game theory

- **Game theory** deals with systems of interacting agents where the outcome for an agent depends on the actions of all the other agents
  - Applied in sociology, politics, economics, biology, and, of course, AI
- **Agent design**: determining the best strategy for a rational agent in a given game
- **Mechanism design**: how to set the rules of the game to ensure a desirable outcome
Modelling behaviour

Game theory in practice

Computing: Software that models human behaviour can make forecasts, outfox rivals and transform negotiations

Sep 3rd 2011 | from the print edition

http://www.economist.com/node/21527025
You can think of John Hegeman as Facebook's chief economist. He spends his days thinking about the economics of Facebook advertising.

That's an enormous thing. Facebook pulled in $4.04 billion in the second quarter of this year. And the overall economy of Facebook advertising, as Hegeman describes it, is far larger. Advertising, you see, is very much a part of everything else on the world's largest social network. Hegeman doesn't just think about ads. He thinks about how ads fit with the rest of Facebook.

When he joined Facebook in 2007, after getting a master's in economics at Stanford University, Hegeman helped build the online auction that drives the company's advertising system. Auctions are the standard way that online services accept ads from advertisers and place them on web pages and inside smartphone apps. That's what Google uses with AdWords, the system that serves up all those ads when you look for stuff on the company's Internet search engine. Advertisers bid (in dollars) for placement on the results page when you key in a particular word or group of words. But in building Facebook's advertising system, Hegeman and team took online auctions in a new direction.

Simultaneous single-move games

- Players must choose their actions at the same time, without knowing what the others will do
  - Form of partial observability

**Normal form** representation:

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rock</strong></td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td><strong>Paper</strong></td>
<td>1,-1</td>
<td>0,0</td>
</tr>
<tr>
<td><strong>Scissors</strong></td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td></td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
</tbody>
</table>

*(Player 1’s utility is listed first)*

Is this a zero-sum game?
Prisoner’s dilemma

- Two criminals have been arrested and the police visit them separately
- If one player testifies against the other and the other refuses, the one who testified goes free and the one who refused gets a 10-year sentence
- If both players testify against each other, they each get a 5-year sentence
- If both refuse to testify, they each get a 1-year sentence

<table>
<thead>
<tr>
<th>Alice: Testify</th>
<th>Alice: Refuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bob: Testify</td>
<td></td>
</tr>
<tr>
<td>Bob: Refuse</td>
<td></td>
</tr>
</tbody>
</table>
Prisoner’s dilemma

- Alice’s reasoning:
  - Suppose Bob testifies. Then I get 5 years if I testify and 10 years if I refuse. So I should testify.
  - Suppose Bob refuses. Then I go free if I testify, and get 1 year if I refuse. So I should testify.

- **Dominant strategy**: A strategy whose outcome is better for the player regardless of the strategy chosen by the other player

<table>
<thead>
<tr>
<th></th>
<th>Alice: Testify</th>
<th>Alice: Refuse</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bob: Testify</td>
<td>-5, -5</td>
<td>-10, 0</td>
</tr>
<tr>
<td>Bob: Refuse</td>
<td>0, -10</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>
Prisoner’s dilemma

- **Nash equilibrium**: A pair of strategies such that no player can get a bigger payoff by switching strategies, provided the other player sticks with the same strategy
  - (Testify, testify) is a *dominant strategy equilibrium*

- **Pareto optimal outcome**: It is impossible to make one of the players better off without making another one worse off

- In a non-zero-sum game, a Nash equilibrium is not necessarily Pareto optimal!
Recall: Multi-player, non-zero-sum game

![Game Tree Diagram]
Prisoner’s dilemma in real life

- Price war
- Arms race
- Steroid use
- Diner’s dilemma
- Collective action in politics

http://en.wikipedia.org/wiki/Prisoner’s_dilemma
Prisoner’s Dilemma in cartoons

Prisoner’s Dilemma in cartoons

https://xkcd.com/1016/
Prisoner’s Dilemma in cartoons

http://www.smbc-comics.com/comic/parenting-game-theory
Is there a way out of Prisoner’s Dilemma?

• Iterated Prisoner’s Dilemma
  – If the number of rounds is fixed and known in advance, and if your strategy cannot affect your opponent’s, the equilibrium strategy is still to defect
  – But if the number of rounds is unknown, and if opponents can respond to each other, things become interesting!

• Tournaments of iterated Prisoner’s Dilemma
  – Video
  – Axelrod’s 1980 tournament
  – LessWrong tournaments: 2011, 2014
  – Training agents by reinforcement learning
Stag hunt

- Is there a dominant strategy for either player?
- Is there a Nash equilibrium?
  - (Stag, stag) and (hare, hare)
- Model for cooperative activity

<table>
<thead>
<tr>
<th>Hunter 2:</th>
<th>Hunter 1:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag</td>
<td>2,2</td>
</tr>
<tr>
<td>Hare</td>
<td>0,1</td>
</tr>
</tbody>
</table>
Prisoner’s dilemma vs. stag hunt

<table>
<thead>
<tr>
<th>Prisoner’s dilemma</th>
<th>Stag hunt</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cooperate</strong></td>
<td><strong>Cooperate</strong></td>
</tr>
<tr>
<td><strong>Cooperate</strong></td>
<td>Win – win</td>
</tr>
<tr>
<td>Lose big – win big</td>
<td>Lose – lose</td>
</tr>
<tr>
<td><strong>Defect</strong></td>
<td>Lose big – win big</td>
</tr>
</tbody>
</table>

Players can gain by defecting unilaterally

Players lose by defecting unilaterally
Game of Chicken

- Is there a dominant strategy for either player?
- Is there a Nash equilibrium?
  - (Straight, chicken) or (chicken, straight)
- Anti-coordination game: it is mutually beneficial for the two players to choose different strategies
  - Model of escalated conflict in humans and animals (hawk-dove game)
- How are the players to decide what to do?
  - Pre-commitment or threats
  - Different roles: the “hawk” is the territory owner and the “dove” is the intruder, or vice versa

Game of Chicken in the movies

Rebel without a cause

Footloose
Mixed strategy equilibria

• **Mixed strategy**: a player chooses between the moves according to a probability distribution

• Suppose each player chooses S with probability $1/10$. Is that a Nash equilibrium?

• Consider payoffs to P1 while keeping P2’s strategy fixed
  – The payoff of P1 choosing S is $(1/10)(-10) + (9/10)1 = -1/10$
  – The payoff of P1 choosing C is $(1/10)(-1) + (9/10)0 = -1/10$
  – Can P1 change their strategy to get a better payoff?
  – Same reasoning applies to P2
Finding mixed strategy equilibria

- Expected payoffs for P1 given P2’s strategy:
  - P1 chooses S: \( q(-10) + (1-q)1 = -11q + 1 \)
  - P1 chooses C: \( q(-1) + (1-q)0 = -q \)

- In order for P2’s strategy to be part of a Nash equilibrium, P1 has to be indifferent between its two actions:
  - \(-11q + 1 = -q\) or \(q = 1/10\)
  - Similarly, \(p = 1/10\)
Existence of Nash equilibria

- Any game with a finite set of actions has at least one Nash equilibrium (which may be a mixed-strategy equilibrium)
- If a player has a dominant strategy, there exists a Nash equilibrium in which the player plays that strategy and the other player plays the best response to that strategy
- If both players have strictly dominant strategies, there exists a Nash equilibrium in which they play those strategies
Computing Nash equilibria

- For a two-player zero-sum game, simple linear programming problem
- For non-zero-sum games, the algorithm has worst-case running time that is exponential in the number of actions
- For more than two players, and for sequential games, things get pretty hairy
Nash equilibria and rational decisions

• If a game has a *unique* Nash equilibrium, it will be adopted if each player
  – is rational and the payoff matrix is accurate
  – doesn’t make mistakes in execution
  – is capable of computing the Nash equilibrium
  – believes that a deviation in strategy on their part will not cause the other players to deviate
  – there is *common knowledge* that all players meet these conditions

http://en.wikipedia.org/wiki/Nash_equilibrium
Continuous actions: Ultimatum game

• Alice and Bob are given a sum of money $S$ to divide
  – Alice picks $A$, the amount she wants to keep for herself
  – Bob picks $B$, the smallest amount of money he is willing to accept
  – If $S - A \geq B$, Alice gets $A$ and Bob gets $S - A$
  – If $S - A < B$, both players get nothing

• What is the Nash equilibrium?
  – Alice offers Bob the smallest amount of money he will accept: $S - A = B$
  – Alice and Bob both want to keep the full amount: $A = S$, $B = S$ (both players get nothing)

• How would humans behave in this game?
  – If Bob perceives Alice’s offer as unfair, Bob will be likely to refuse
  – Is this rational?
    • Maybe Bob gets some positive utility for “punishing” Alice?

Sequential/repeated games: Chain store paradox

- A monopolist has branches in 20 towns and faces 20 competitors successively
- How should the monopolist respond to competitors going “in”?

https://en.wikipedia.org/wiki/Chainstore_paradox
Mechanism design (inverse game theory)

• Assuming that agents pick rational strategies, how should we design the game to achieve a socially desirable outcome?
• We have multiple agents and a center that collects their choices and determines the outcome.
Auctions

• Goals
  – Maximize revenue to the seller
  – *Efficiency*: make sure the buyer who values the goods the most gets them
  – Minimize transaction costs for buyer and sellers
Ascending-bid auction

• What’s the optimal strategy for a buyer?
  – Bid until the current bid value exceeds your *private value*

• Usually revenue-maximizing and efficient, unless the reserve price is set too low or too high

• Disadvantages
  – Collusion
  – Lack of competition
  – Has high communication costs
Sealed-bid auction

- Each buyer makes a single bid and communicates it to the auctioneer, but not to the other bidders
  - Simpler communication
  - More complicated decision-making: the strategy of a buyer depends on what they believe about the other buyers
  - Not necessarily efficient

- **Sealed-bid second-price auction:** the winner pays the price of the second-highest bid
  - Let $V$ be your private value and $B$ be the highest bid by any other buyer
  - If $V > B$, your optimal strategy is to bid above $B$ – in particular, bid $V$
  - If $V < B$, your optimal strategy is to bid below $B$ – in particular, bid $V$
  - Therefore, your dominant strategy is to bid $V$
  - This is a **truth revealing** mechanism
Application: Real-time bidding for Internet ads

http://tutorial.computational-advertising.org/final-slides.pdf
Application: Real-time bidding for Internet ads

Buying advertising in Real Time - an Example

0.04 Sec – User ABC clicks on a URL and the publisher’s content begins to load in browser
0.08 Sec – Publisher asks its ad server if and ad is available. If no ad, server asks Ad Exchange
0.10 Sec – Ad Exchange federates ad request to multiple demand side platforms (DSPs), the technology for buying media
0.12 Sec – Ad Exchange sends each DSP User ABCs anonymous profile website category and page ad safety information
0.125 Sec – Each DSP overlays advertiser targeting and budget rules, and applies third-party data
0.13 Sec – Each DSP algorithm evaluates and computes optimal bid for advertiser
0.14 Sec – Each DSP responds to Ad Exchange
0.18 Sec – Ad Exchange runs a second-price auction and selects winning bid from DSP responses
0.19 Sec – Ad Exchange sends price and ad from winning bid to publisher’s ad server
0.23 Sec – Publisher’s ad server tells browser which ad to display
0.31 Sec – Advertiser’s ad server sends winning ad to browser
0.36 Sec – Browser displays web page including winning ad, and signals to winning DSP the ad was viewed

Dollar auction

• A dollar bill is auctioned off to the highest bidder, but the second-highest bidder has to pay the amount of his last bid
  – Player 1 bids 1 cent
  – Player 2 bids 2 cents
  – ...
  – Player 2 bids 98 cents
  – Player 1 bids 99 cents
    • If Player 2 passes, he loses 98 cents, if he bids $1, he might still come out even
  – So Player 2 bids $1
    • Now, if Player 1 passes, he loses 99 cents, if he bids $1.01, he only loses 1 cent
  – ...

• What went wrong?
  – When figuring out the expected utility of a bid, a rational player should take into account the future course of the game

• What if Player 1 starts by bidding 99 cents?
Dollar auction

- A dollar bill is auctioned off to the highest bidder, but the second-highest bidder has to pay the amount of his last bid

- Dramatization: https://www.youtube.com/watch?v=pA-SNscNADk
Regulatory mechanism design: Tragedy of the commons

• States want to set their policies for controlling emissions
  – Each state can reduce their emissions at a cost of -10
    or continue to pollute at a cost of -5
  – If a state decides to pollute, -1 is added to the utility of every other state

• What is the dominant strategy for each state?
  – Continue to pollute
  – Each state incurs cost of -5-49 = -54
  – If they all decided to deal with emissions, they would incur a cost of only -10 each

• Mechanism for fixing the problem:
  – Tax each state by the total amount by which they reduce the global utility (externality cost)
  – This way, continuing to pollute would now cost -54
Review: Game theory

- Normal form representation of a game
- Dominant strategies
- Nash equilibria
- Pareto optimal outcomes
- Pure strategies and mixed strategies
- Examples of games
  - Prisoner’s Dilemma, Stag Hunt, Chicken
- Mechanism design
  - Regulatory mechanism design